Incremental Cycle Detection

&Topological Ordering

&Strong Components

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Given a directed graph, does it contain a cycle?

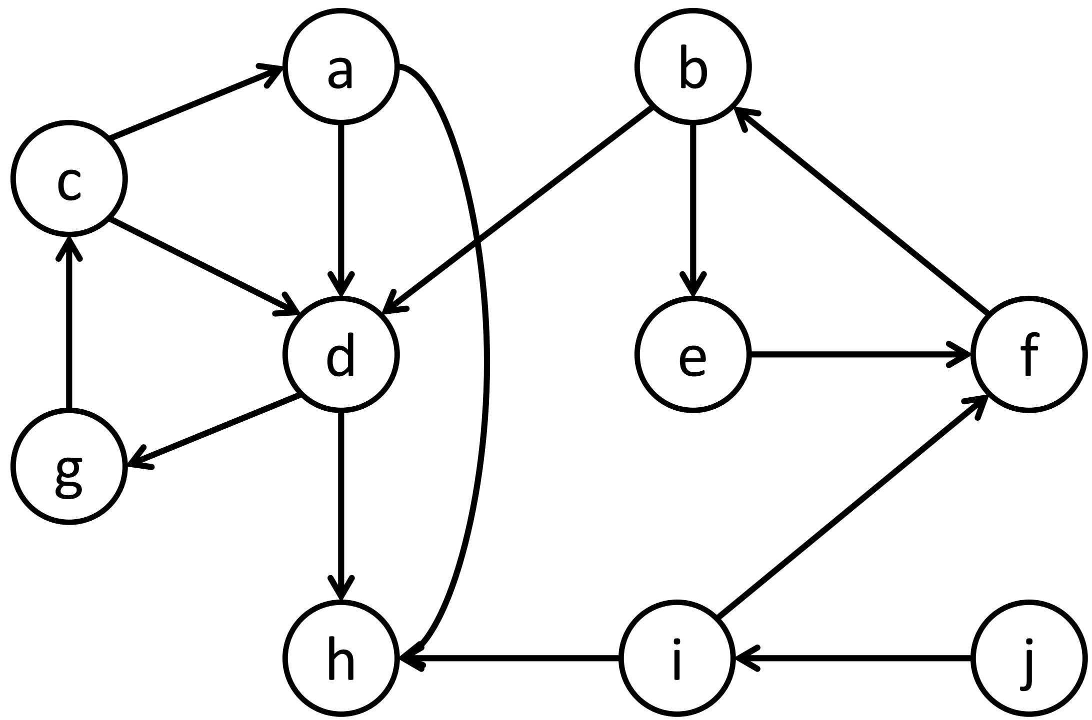
(Given a set of dependencies, do they contain a circularity?)

Given a digraph, order its vertices *topologically*: if (v, w) is an arc, $v < w$.

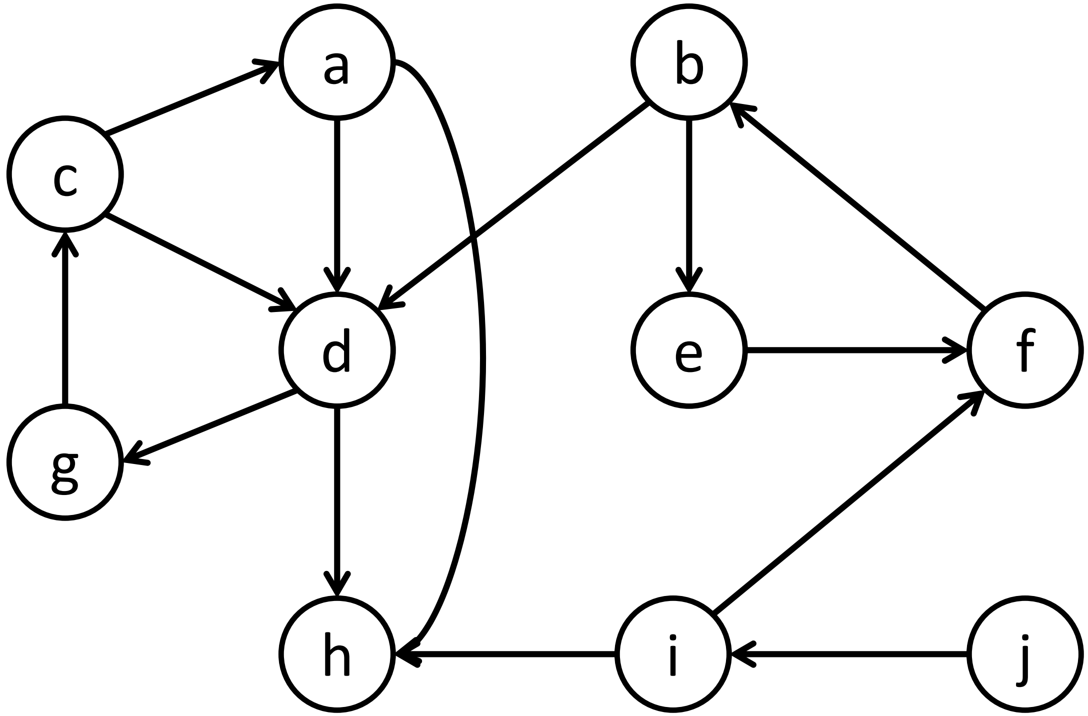
Given a digraph, find its strong components: maximal sets of mutually reachable vertices.

A directed graph

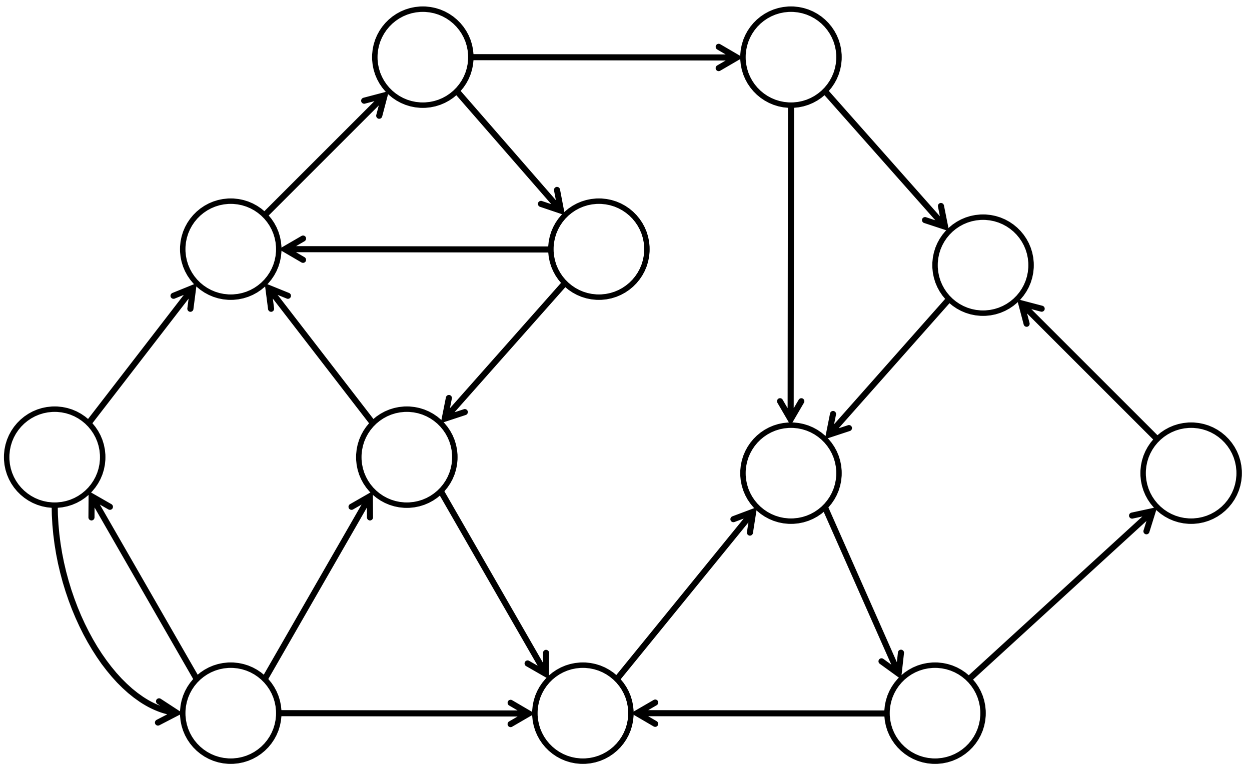
Is there a cycle?

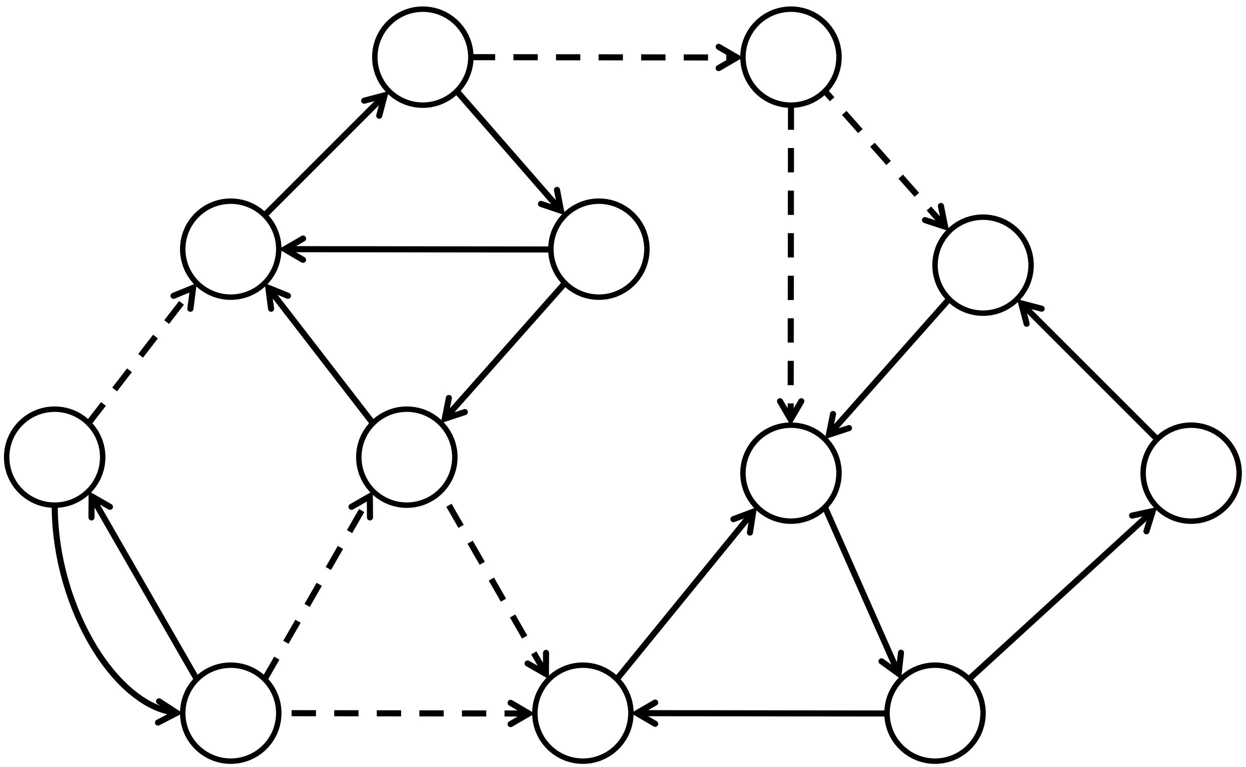


Yes: a, d, g, c



Strong components





A digraph is acyclic if and only if it can be topologically ordered.

Two linear-time algorithms:

1) Repeatedly delete and number sources (vertices with no incoming arcs)

2) Do a depth-first search, number vertices in reverse postorder (finishing order)

Method (1) finds all possible topological orderings, method (2) finds only some (in general)

What if digraph is dynamic?

Incremental: No arcs initially, arcs added one at a time (or in batches)

Decremental: Arcs deleted one at a time (or in batches)

(*Fully*) *Dynamic*: Arcs added or deleted

Incremental cycle detection

Extends to maintenance of a topological ordering, and to maintenance of strong components.

Goal: Beat $O(m^2)$ time for m arc additions.

Assume vertex set fixed, of size n.

Previous Results

String of better and better methods led to a time bound of O($m^{3/2}$).

The algorithm uses two-way search, fancy data structures, and median-finding or randomization.

[B. Haeupler, T. Kavitha, R. Mathew, S. Sen, & R. Tarjan, “Faster algorithms for incremental topological ordering,” ICALP 2008, pp 421-433; revision to appear in ACM Trans. Alg.]

Our Results

A simple O(Δm)-time algorithm, where

$\Delta = \min\{m^{1/2}, n^{2/3}\}$

An O($n^2 \log n$)-time algorithm ($m < n^2$)

I will describe the first; for the second see [M. Bender, J. Fineman, & S. Gilbert, "A new approach to incremental topological ordering," SODA 2009, pp. 1108-1115, revised extension submitted to JACM (with added co-author)]

How to handle arc additions?

If a topological order is maintained, and new arc (v, w) has $v < w$, nothing to do, but if $v > w$, must search forward from w and/or backward from v until finding a path from w to v (giving a cycle) or finding a way to restore topological order.

This is how previous algorithms worked.

We maintain only a partial topological order, in the form of a vertex numbering.

Each vertex v has a positive integer *level* $k(v)$, between 1 and Δ, initially 1, only increasing.

Weak topological numbering:

if (v, w) is an arc, $k(v) \le k(w)$.

We use asymmetric two-way search.

To add (v, w)

If $k(v) < k(w)$ then add (v, w) else

{search backward from v in same level until

either the search visits w: stop (cycle)

or the search traverses at least Δ arcs:

{ increase $k(w)$ to $k(v) + 1$; search forward}

or the search finishes:

if $k(\mathrm{v}) > k(w)$ then

{increase $k(w)$ to $k(v)$; search forward};

add (v, w)}

Search forward: traverse all arcs out of vertices whose level increases, starting with w; when traversing an arc to a vertex x such that k(x) < k(w), increase k(x) to k(w) and add arcs out of x to those to be traversed; if search visits v, stop (cycle).

Implementation: For each vertex, maintain the set of all outgoing arcs, and the set of incoming arcs from vertices on the same level.

Correctness & Analysis

Correctness is straightforward: algorithm will detect a cycle if and only if there is one; if still acyclic, the level increases restore weak topological numbering.

Efficiency: Backward searches take O(Δm) time.

Forward searches take O(Km) time, where K is maximum level.

Lemma: $K \leq \Delta + 2$.

Proof: Fix a topological order just before the arc addition (if any) that creates a cycle. Consider the levels at this time. (The last arc addition can only increase the maximum level by one.) For any level $k > 1$, let w be the first vertex on level k. For w to have been promoted to level k, a backward search must have visited Δ arcs and $\Delta^{1/2}$ vertices on level $k - 1$, which remain on level $k - 1$. These sets of arcs and vertices are disjoint for different k.

Better?

$O(nm^{1/2})$?